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Master Thesis

On universal properties of gapless frustration-free quantum systems

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List of Papers

Chapter 2

- **R. Masaoka**, T. Soejima, and H. Watanabe, Physical Review B 110, 195140 (2024).

Chapter 3

- **R. Masaoka**, S. Ono, H. C. Po, and H. Watanabe, arXiv:2503.12879 (2025).
- S. Ono, **R. Masaoka**, H. Watanabe, and H. C. Po, arXiv:2503.14312 (2025).

Chapter 4

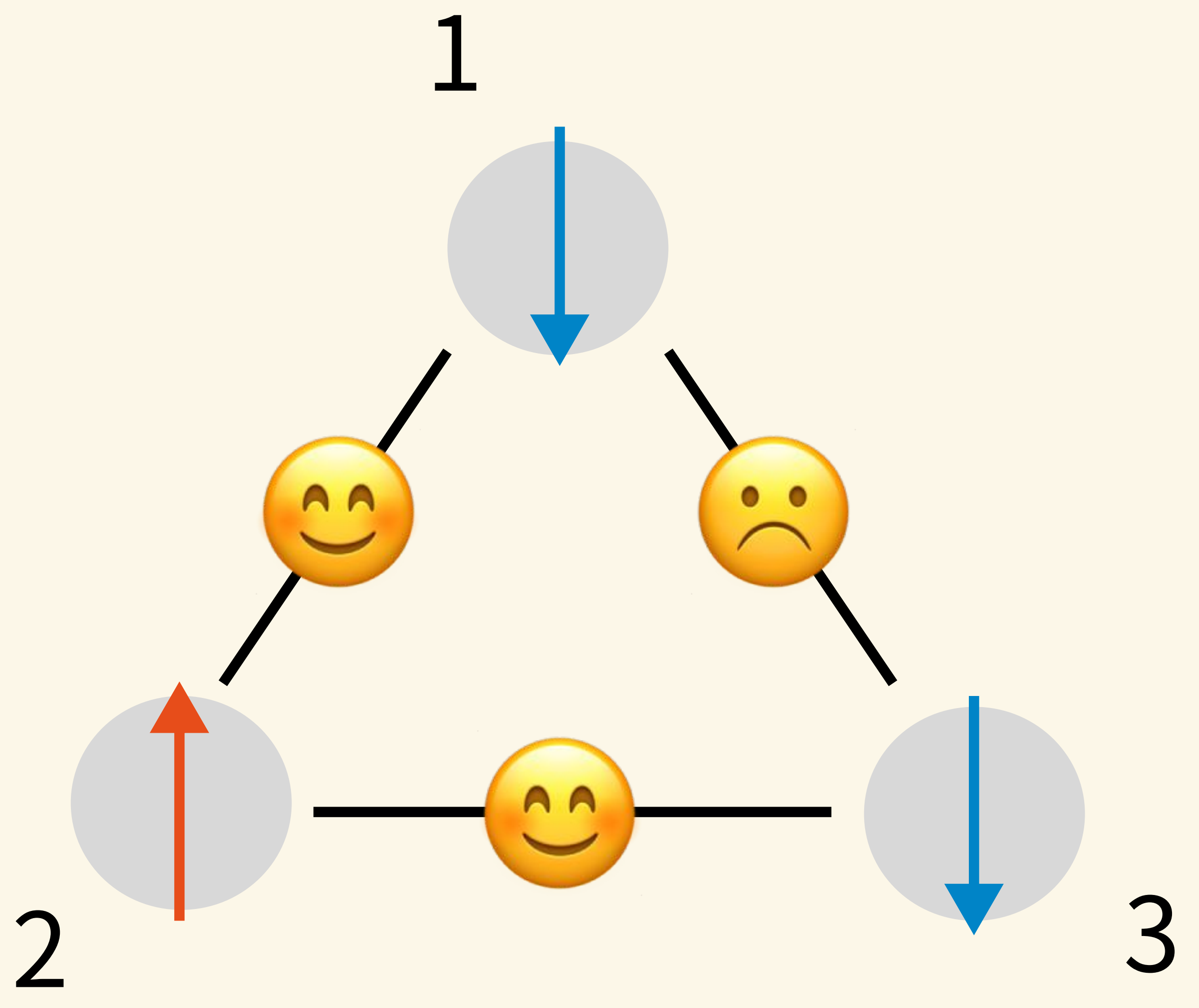
- **R. Masaoka**, T. Soejima, and H. Watanabe, Physical Review X 15, 041050 (2025).
- **R. Masaoka**, T. Soejima, and H. Watanabe, Journal of Statistical Physics 192, 76 (2025).

(Related but not included in the thesis: **R. Masaoka**, arXiv:2511.16496 (2025).)

What is Frustration?

Antiferromagnetic Ising model

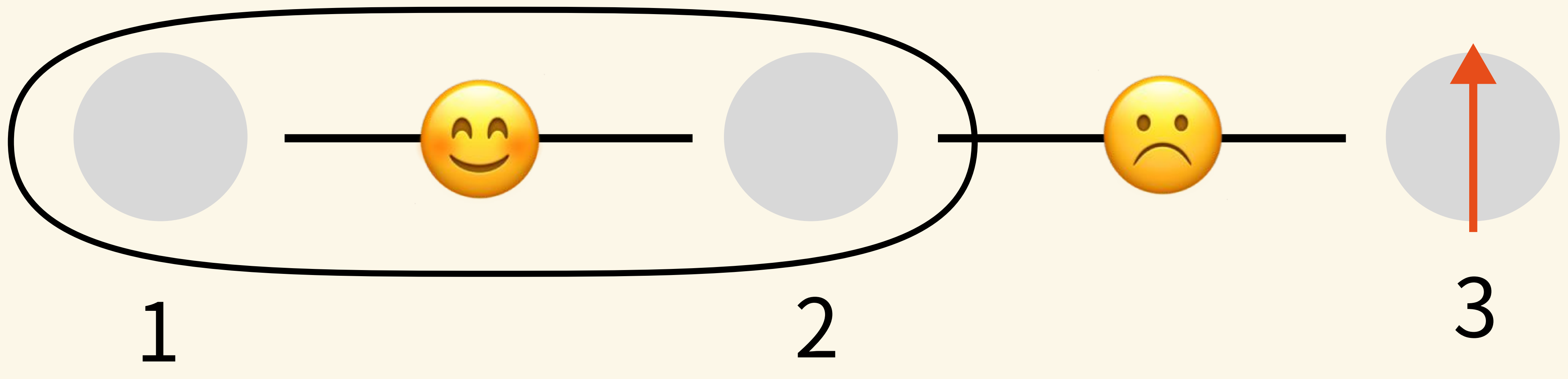
$$\hat{H} = \hat{Z}_1\hat{Z}_2 + \hat{Z}_2\hat{Z}_3 + \hat{Z}_3\hat{Z}_1$$



Geometric frustration

Antiferromagnetic spin-1/2 Heisenberg model

$$\hat{H} = \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_2 \cdot \hat{\boldsymbol{S}}_3$$



$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes |\uparrow\rangle$$

Quantum frustration

Our focus: Frustration-free quantum systems

Frustration-free quantum systems

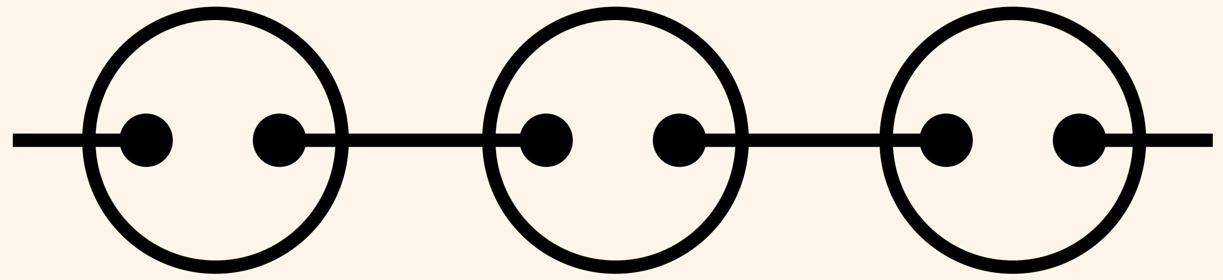
A Hamiltonian is called frustration-free (FF) iff there exist a decomposition of the Hamiltonian

$$\hat{H} = \sum_i \hat{H}_i$$

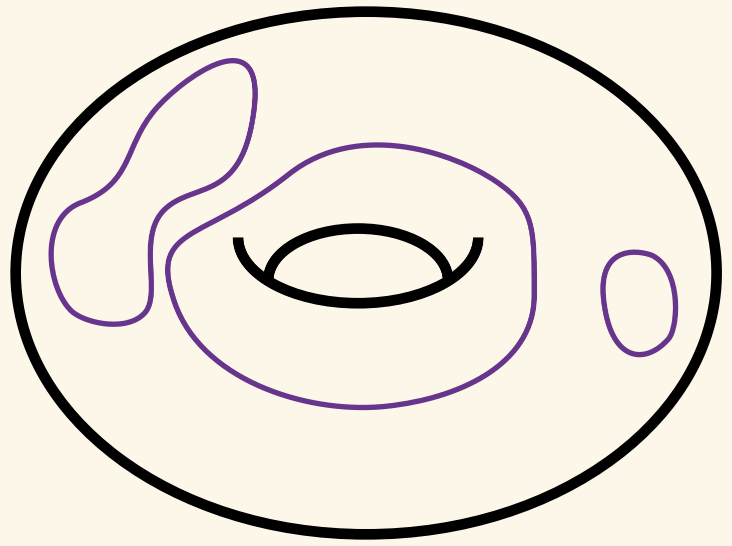
and a ground state $|\mathrm{GS}\rangle$ s.t.

- Each term $\hat{H}_i$ is a local operator.
- The ground state $|\mathrm{GS}\rangle$ minimizes all local terms simultaneously.

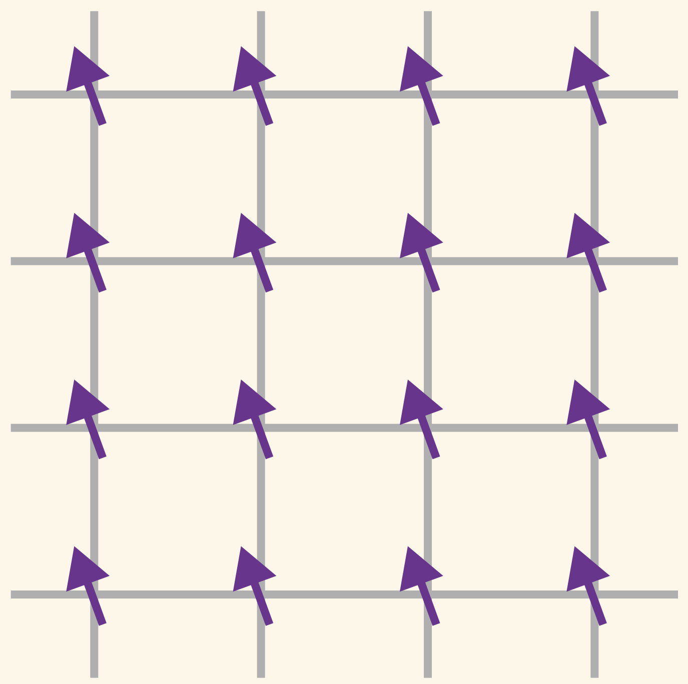
Examples:



AKLT model



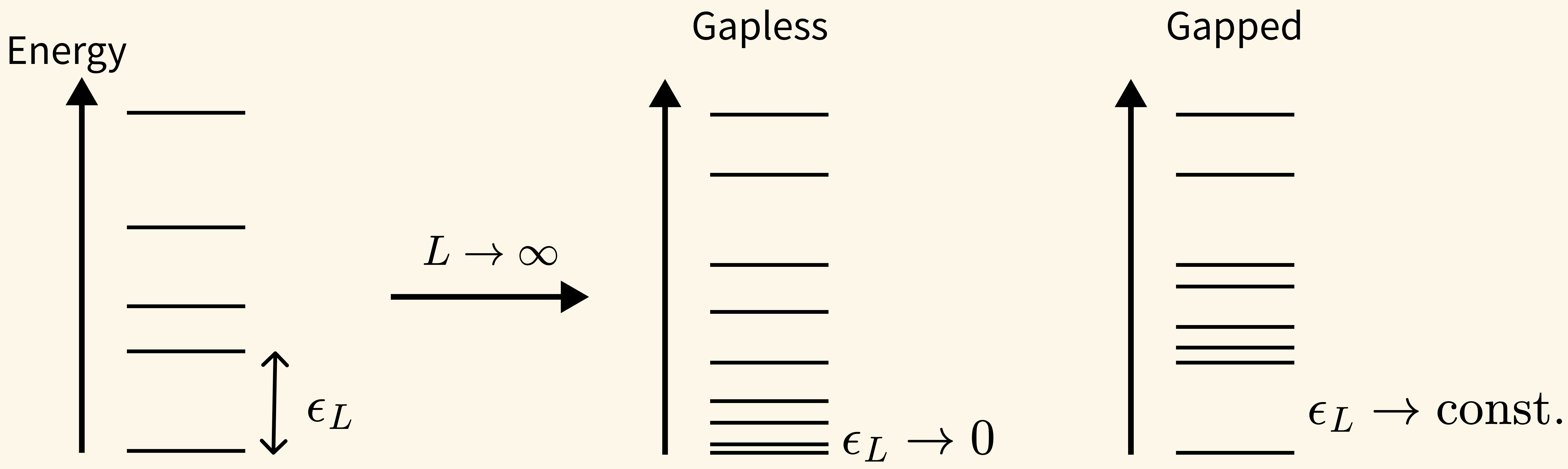
Toric code



Ferromagnetic Heisenberg model

Preliminary: gapped and gapless quantum Hamiltonians

L : Side length of the system



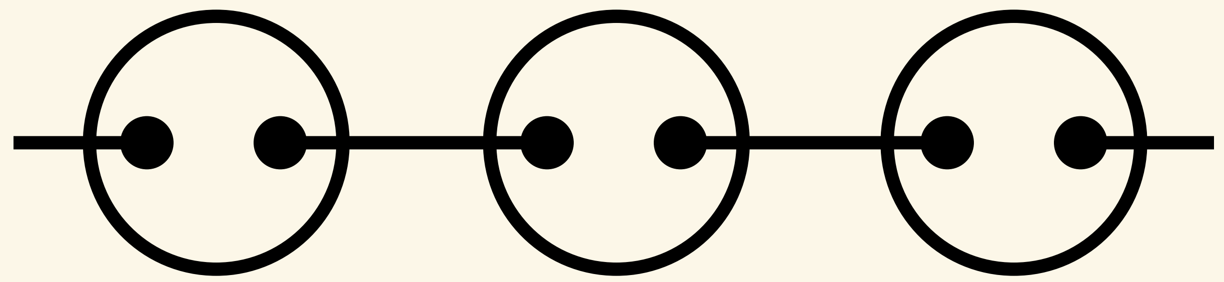
ϵ_L : Finite size gap of $\hat{H}_L$

(We ignore subtleties about ground state degeneracy in this presentation)

Why gapless FF systems are interesting?

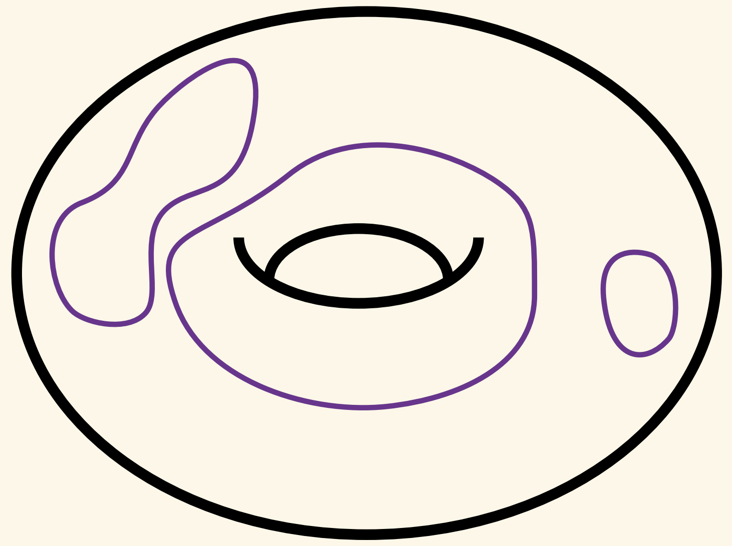
Historically, FF systems have provided many pivotal toy models for gapped quantum phases.

AKLT model



Symmetry protected topological phase

Toric code



Topological order

For gapped quantum phases, frustration-freeness is an artificial condition for theoretical tractability.

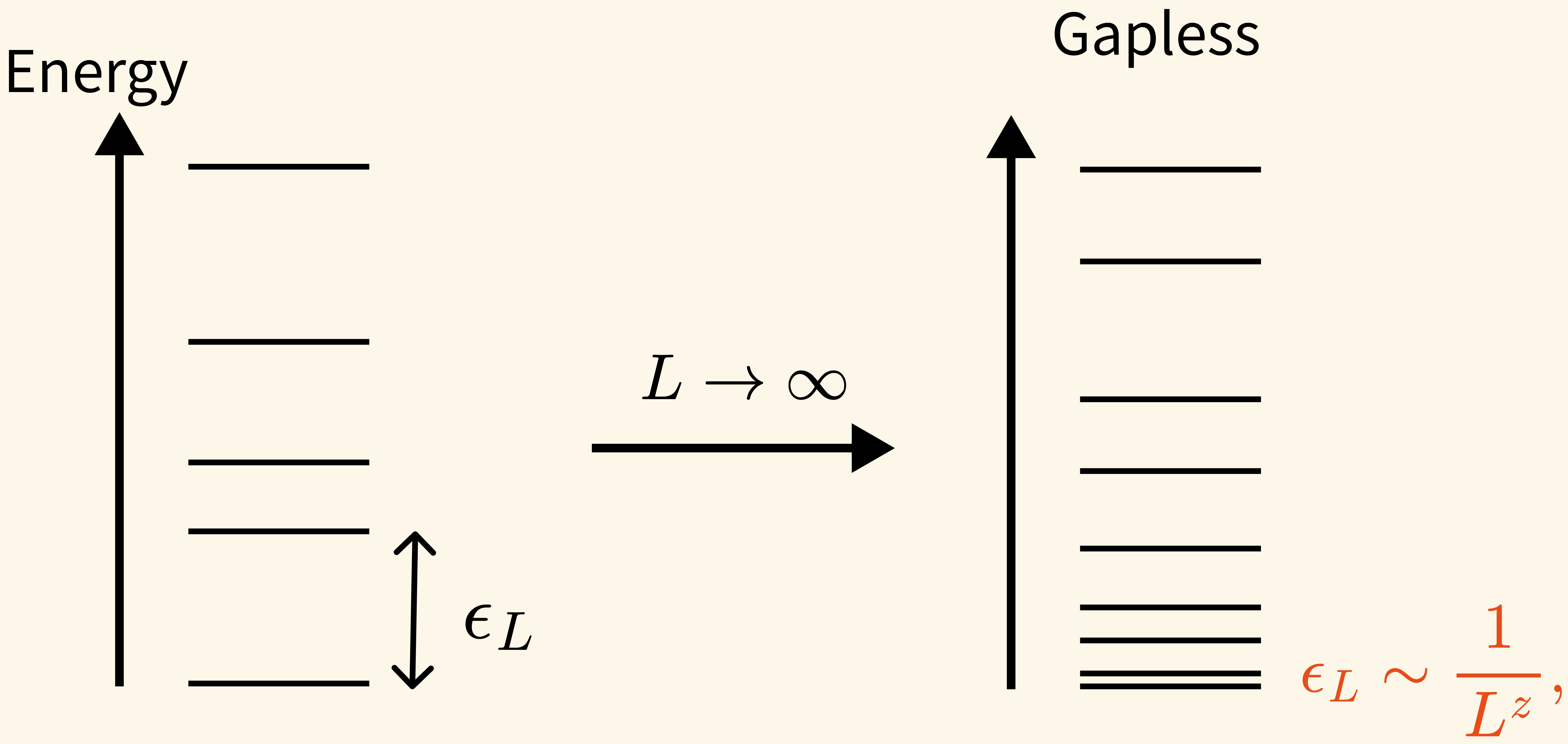
However, for gapless systems, frustration-freeness has an impact on their universal properties!

Prior works

- Conformal quantum critical points $\subset$ gapless FF systems
  Ardonne et al., Ann. Phys. (Amsterdam) 310, 493 (2004).
  Isakov et al., PRB 83, 125114 (2011).
- Nambu-Goldstone modes in FF systems $\subset$ gapless FF systems
  Ogunnaike, Feldmeier, Lee, PRL 131, 220403 (2023).
  Ren, Wang, Fang, PRB 110, 245101 (2024).
- Studies of concrete models $\subset$ gapless FF systems
  Chen, Fradkin, Witczak-Krempa, J. Phys. A 50 464002 (2017).
  Kumer et al, Sci. Rep. 11 1004 (2021).
  Tantivasadakarn et al., SciPost Phys. 14, 012 (2023)
  Saito, Hotta, PRL 132, 166701 (2024).

There have been many observations of nontrivial dynamical exponents in various gapless FF systems!

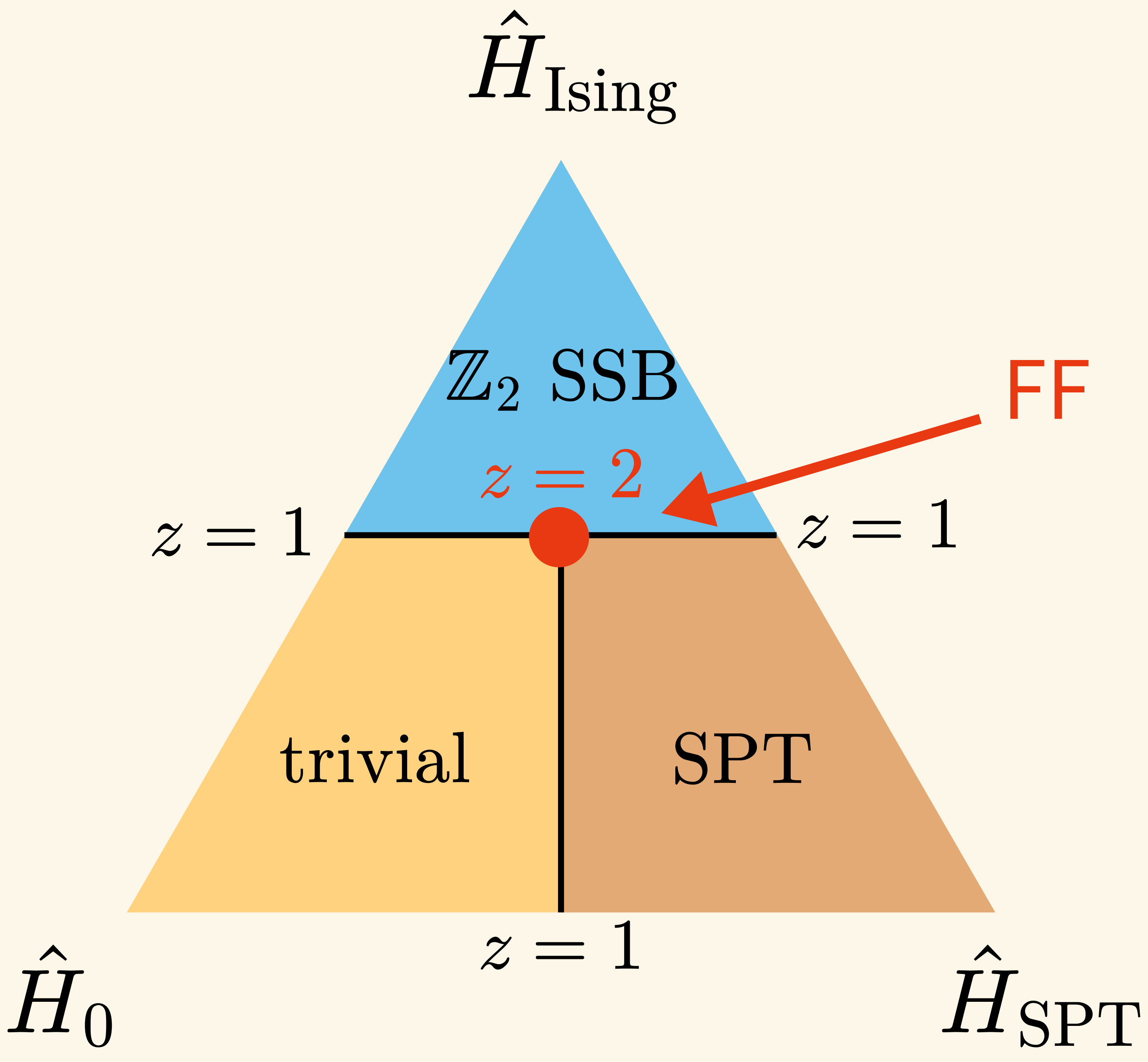
Definition of dynamical exponent z



Typical gapless systems $\to$ $z = 1$
(conformal field theory, fermi liquid)

Example

Kumer et al, Sci. Rep. 11 1004 (2021).
Tantivasadakarn et al., SciPost Phys. 14, 012 (2023)



$$\hat{H} = \alpha \hat{H}_0 + (1-\alpha)\hat{H}_{\text{SPT}} + h\hat{H}_{\text{Ising}}$$

$$\hat{H}_0 = -\sum_{i=1}^{L} \hat{X}_i$$

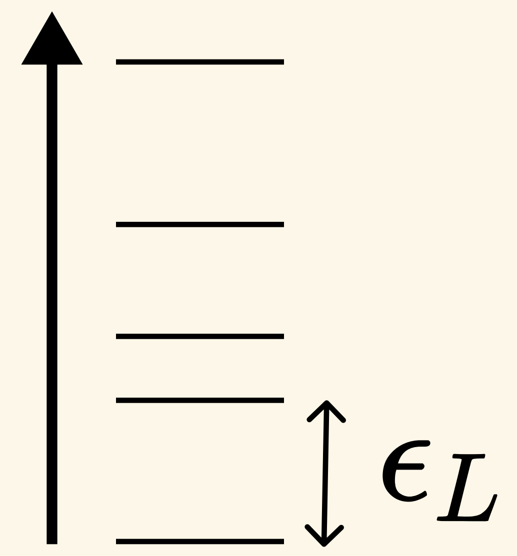
$$\hat{H}_{\text{SPT}} = -\sum_{i=1}^{L} \hat{Z}_{i-1}\hat{X}_i\hat{Z}_i$$

$$\hat{H}_{\text{Ising}} = -\sum_{i=1}^{L} \hat{Z}_i\hat{Z}_{i+1}$$

Two conjectures

RM, Soejima, Watanabe, PRB 110, 195140 (2024).

Definition of dynamical exponent z :

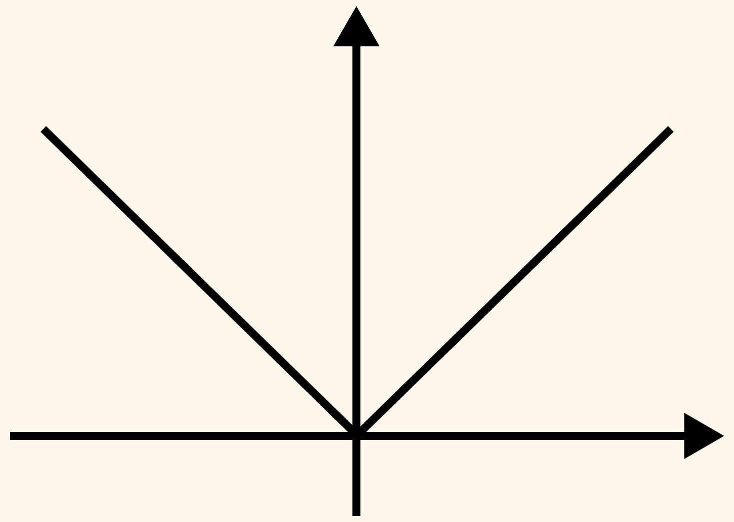


$$\epsilon_L \sim \frac{1}{L^z},$$

Typical gapless systems $\rightarrow z = 1$

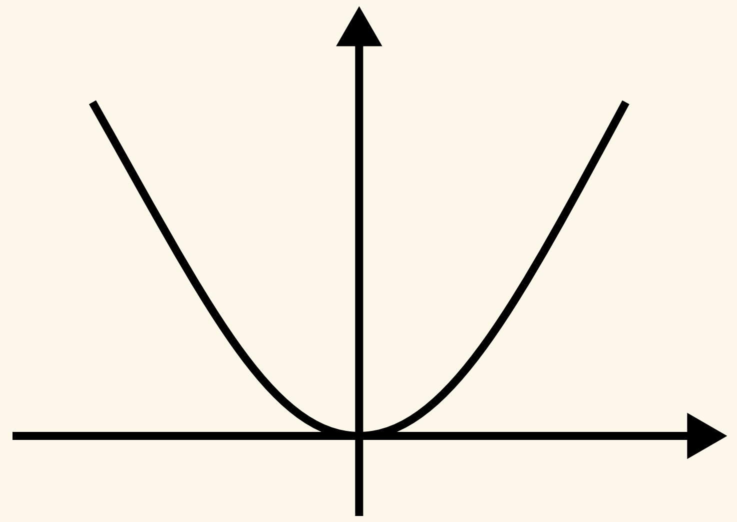
Related dispersion relation: $\omega_k \sim k^z$

Linear dispersion



$z = 1$

FF



$z \geq 2$

Conjecture 1

For gapless FF systems, dynamical exponent satisfies $z \geq 2$

Conjecture 2

Low-energy excitations of gapless FF systems exhibit $\omega_k \sim k^z$ with $z \geq 2$

Our contributions

1. We first formulate unified conjectures on gapless FF systems.

   RM, Soejima, Watanabe, PRB 110, 195140 (2024).

2. We prove them in several settings, including spin-1/2 models and free fermions.

   RM, Soejima, Watanabe, PRB 110, 195140 (2024).
   RM, S. Ono, Po, Watanabe, arXiv:2503.12879 (2025).
   Ono, RM, Watanabe, Po, arXiv:2503.14312 (2025).

3. Assuming power-law correlations, we proved the first conjecture.

   RM, Soejima, Watanabe, PRX 15, 041050 (2025).

4. We apply our results to classical stochastic processes, proving a long-standing empirical fact.

   RM, Soejima, Watanabe, PRX 15, 041050 (2025).
   RM, Soejima, Watanabe, J. Stat. Phys. 192, 76 (2025).

1. Unified conjecture

RM, Soejima, Watanabe, PRB 110, 195140 (2024).

- Conformal quantum critical points
  Ardonne et al., Ann. Phys. (Amsterdam) 310, 493 (2004).
  Isakov et al., PRB 83, 125114 (2011).
- Nambu-Goldstone modes in FF systems
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  Kumer et al, Sci. Rep. 11 1004 (2021).
  Tantivasadakarn et al., SciPost Phys. 14, 012 (2023)
  Saito, Hotta, PRL 132, 166701 (2024).
- Local gap threshold in FF systems
  Gosset, Mozgunov. J. Math. Phys. (N. Y.) 57, 091901 (2016).
  Anshu, PRB 101, 165104 (2020).
  Lemm, Xiang, J. Pys. A 55, 295203 (2022).
- Correlation vs gap in FF systems
  Gosset, Huang, PRL 116, 097202 (2016).

Unifying picture has remained absent!

Conjecture 1

For gapless FF systems, dynamical exponent satisfies $z \geq 2$

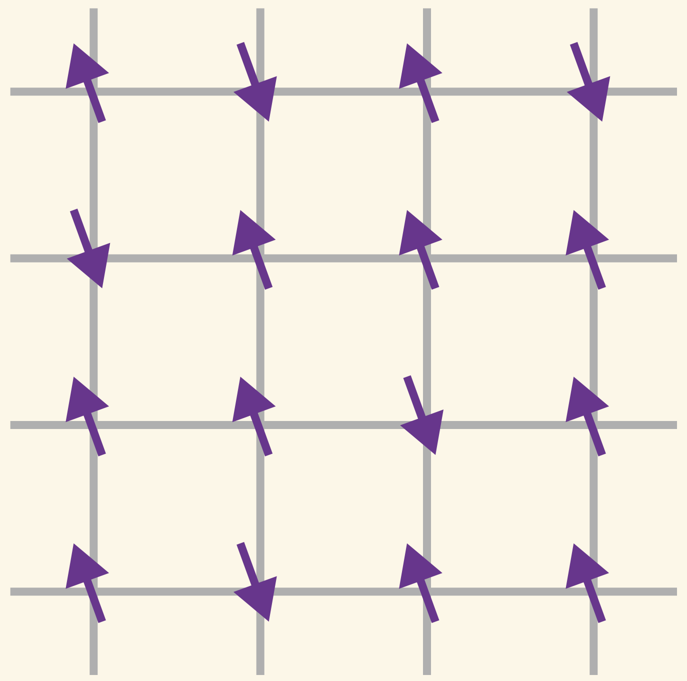
Conjecture 2

Low-energy excitations of gapless FF systems exhibit $\omega_k \sim k^z$ with $z \geq 2$

2-A. Frustration-free spin models

RM, Soejima, Watanabe, PRB 110, 195140 (2024).

We provided a general proof of our conjectures for any spin-1/2 models with nearest-neighbor interactions defined on hypercubic lattices, by using



- Classification of FF spin-1/2 chains with nearest-neighbor interactions.

Bravyi, Gosset, J. Math. Phys. 56, 061902 (2015).

Standard forms: $$\hat{H}_{x,x+1} \propto (\hat{M}\hat{\Pi}^{s=0}_{x,x+1}\hat{M}^{-1})^\dagger(\hat{M}\hat{\Pi}^{s=0}_{x,x+1}\hat{M}^{-1})$$

$$\hat{H}_{x,x+1} = \frac{1}{2}\hat{\mathbb{1}} - \frac{\zeta\hat{s}^+_x\hat{s}^-_{x+1} + \text{h.c.}}{1+|\zeta|^2} - \frac{|\zeta|^2\hat{s}^z_x + \hat{s}^z_{x+1}}{1+|\zeta|^2}$$

- Min-max principle for higher-dimensional generalizations. $\hat{H}' \geq \hat{H} \Rightarrow E'_j \geq E_j$

2-B. Frustration-free free fermion

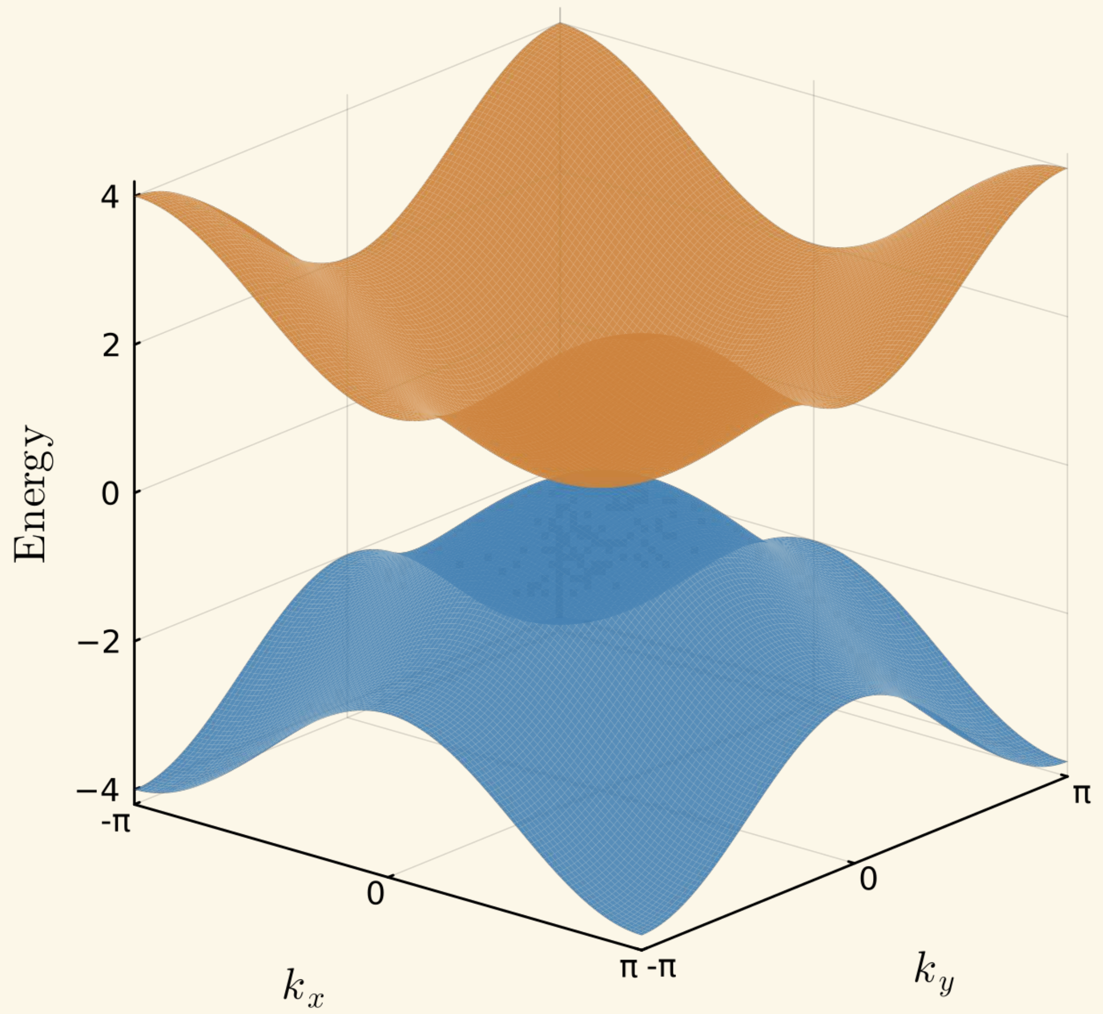
RM, S. Ono, Po, Watanabe, arXiv:2503.12879 (2025).
Ono, RM, Watanabe, Po, arXiv:2503.14312 (2025).

- We established a general framework of frustration-free free fermion systems and derived a necessary and sufficient condition for frustration-freeness.

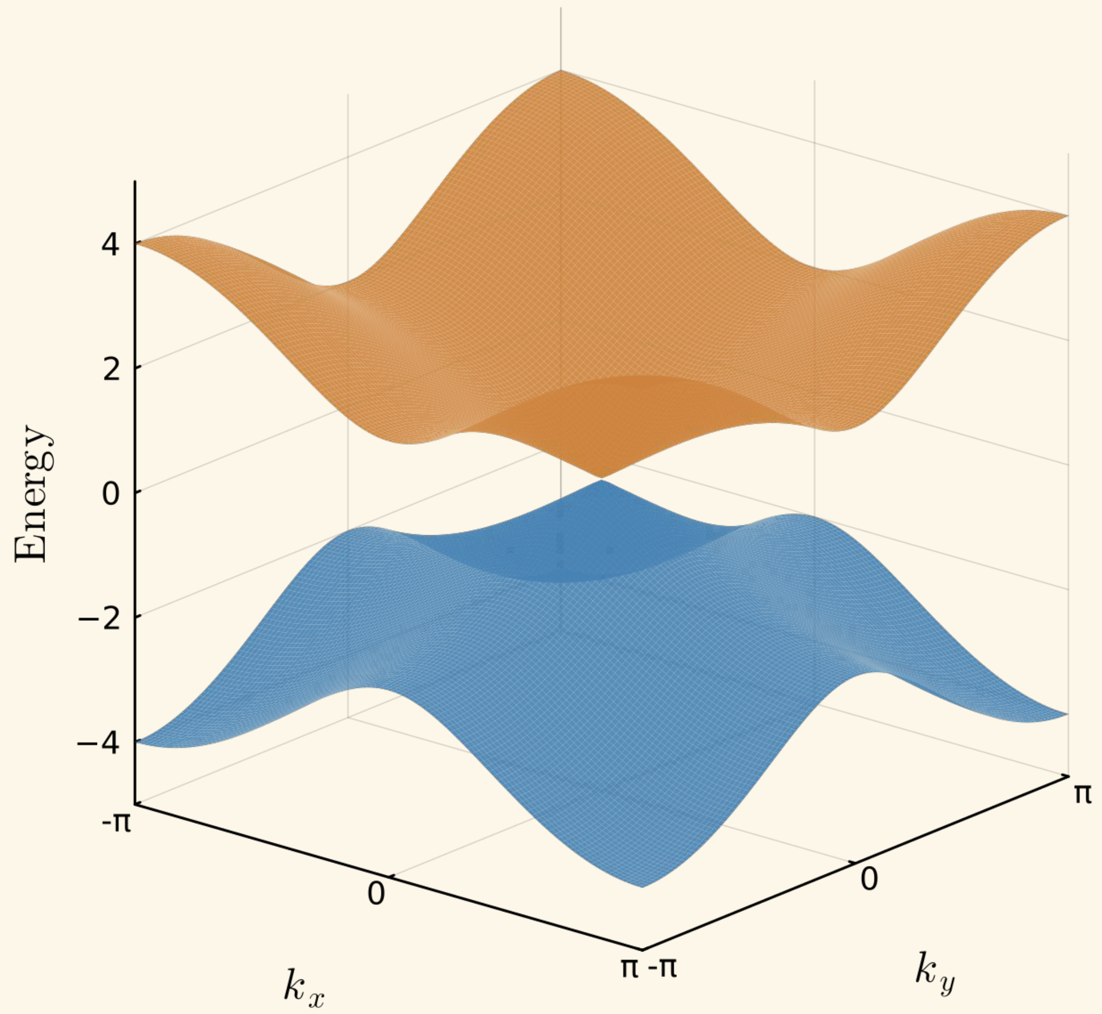
$$\{\hat{\psi}_{\boldsymbol{R}\alpha}, \hat{\phi}^\dagger_{\boldsymbol{R}'\beta}\} = 0, \quad \hat{H}_{\boldsymbol{R}} = \sum_\alpha \mu_\alpha \hat{\psi}^\dagger_{\boldsymbol{R}\alpha}\hat{\psi}_{\boldsymbol{R}\alpha} + \sum_\beta \nu_\beta \hat{\phi}_{\boldsymbol{R}\beta}\hat{\phi}^\dagger_{\boldsymbol{R}\beta}.$$

- Using this framework, we have shown our conjectures.

FF



Frustrated



3. Rigorous bound on dynamical exponent

Theorem. RM, Soejima, Watanabe, PRX 15, 041050 (2025).

FF systems satisfy $z \geq 2$ if their ground states exhibit power-law decaying correlation functions.

Our argument is highly general because we do not assume

- boundary condition
- spatial dimension
- structure of the lattice
- bosonic or fermionic

Our proof is based on the Gosset-Huang inequality. Gosset, Huang, PRL 116, 097202 (2016).

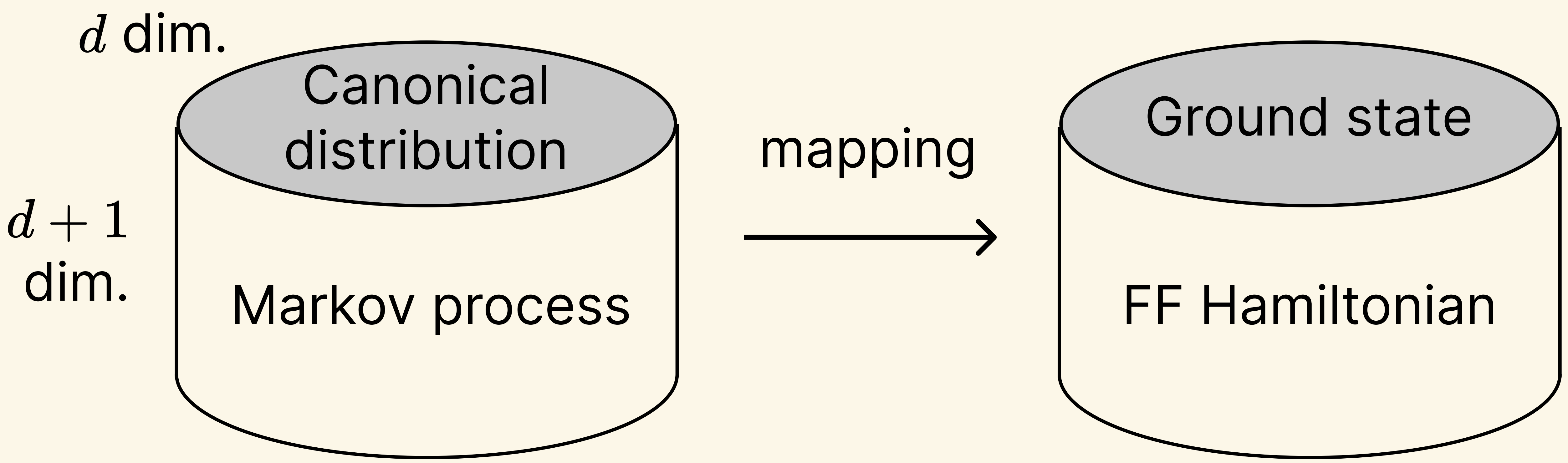
$$\frac{|\langle\Psi|\hat{O}(\hat{\mathbb{1}}-\hat{G})\hat{O}'|\Psi\rangle|}{\|\hat{O}^\dagger\left|\Psi\rangle\|\|\hat{O}'\right|\Psi\rangle\|} \leq 2\exp\left(-\left(\frac{D(\hat{O},\hat{O}')-1}{c-1}-2\right)\sqrt{\frac{\epsilon}{g^2+\epsilon}}\right)$$

We also extended this inequality to expand its applicability.

4. Application to stochastic processes

We apply our proof of $z \geq 2$ to stochastic processes, using an established mapping from Markov processes with detailed balance to FF quantum systems.

Henley, J. Phys. Condens. Matter 16, S891 (2004).
Castelnovo, Chamon, Mudry, Pujol, Ann. Phys. (Amsterdam) 318, 316 (2005).



Theorem. RM, Soejima, Watanabe, PRX 15, 041050 (2025).

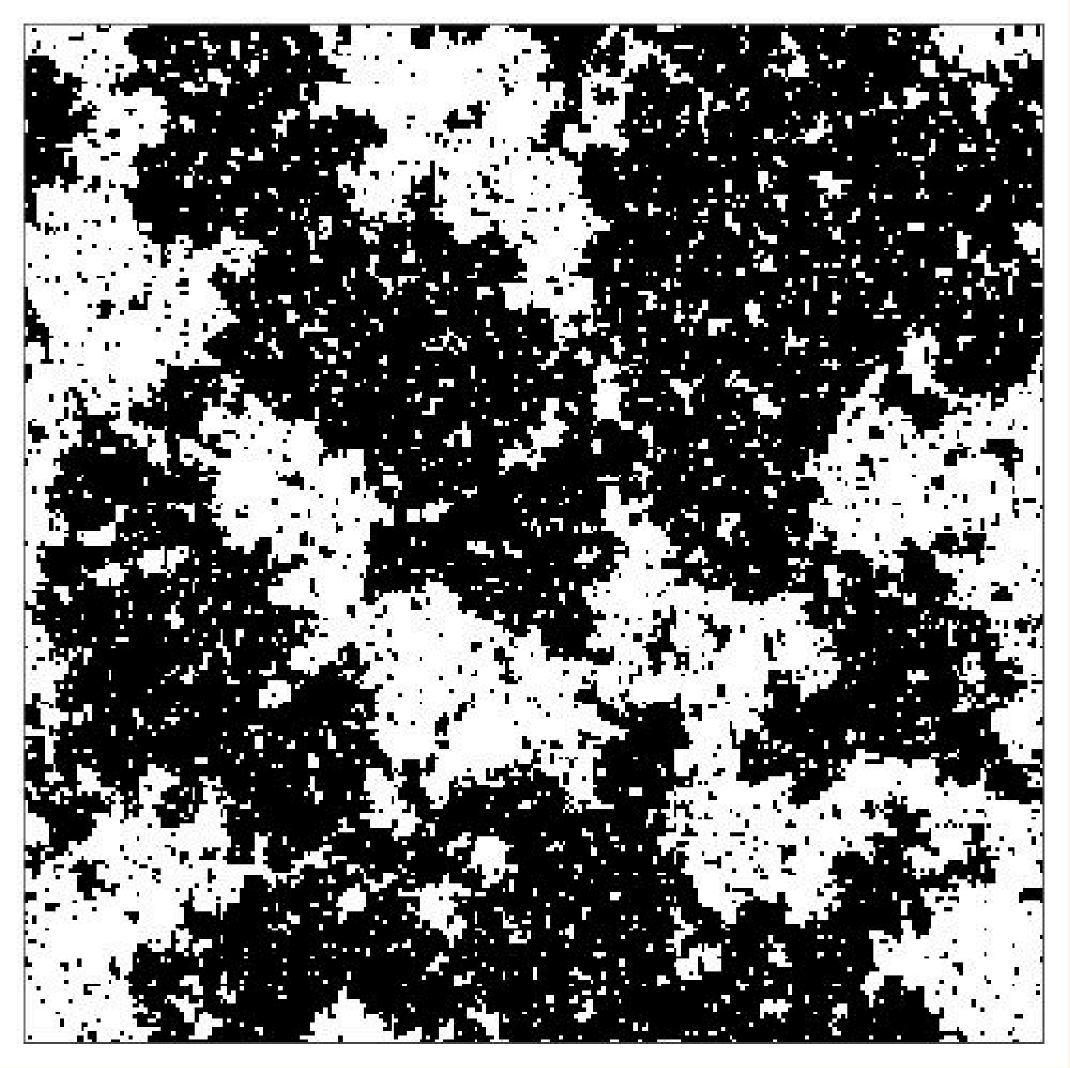
Dynamical exponents of local Markov processes with detailed balance relaxing to critical equilibrium states satisfy $z \geq 2$.

Example: kinetic Ising model

RM, Soejima, Watanabe, J. Stat. Phys. 192, 76 (2025).

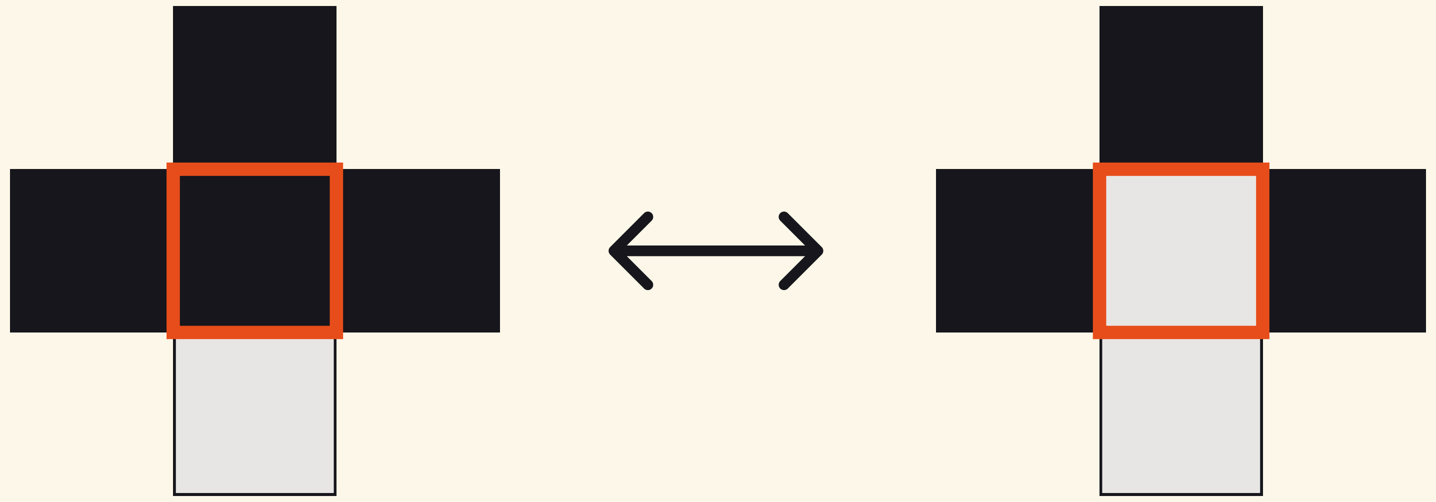
Ising model: $p_{\mathrm{eq}}(\boldsymbol{\sigma}) = \frac{1}{Z} e^{-\beta_c E(\boldsymbol{\sigma})}, \; E(\boldsymbol{\sigma}) = -\sum_{(i,j)} \sigma_i \sigma_j$

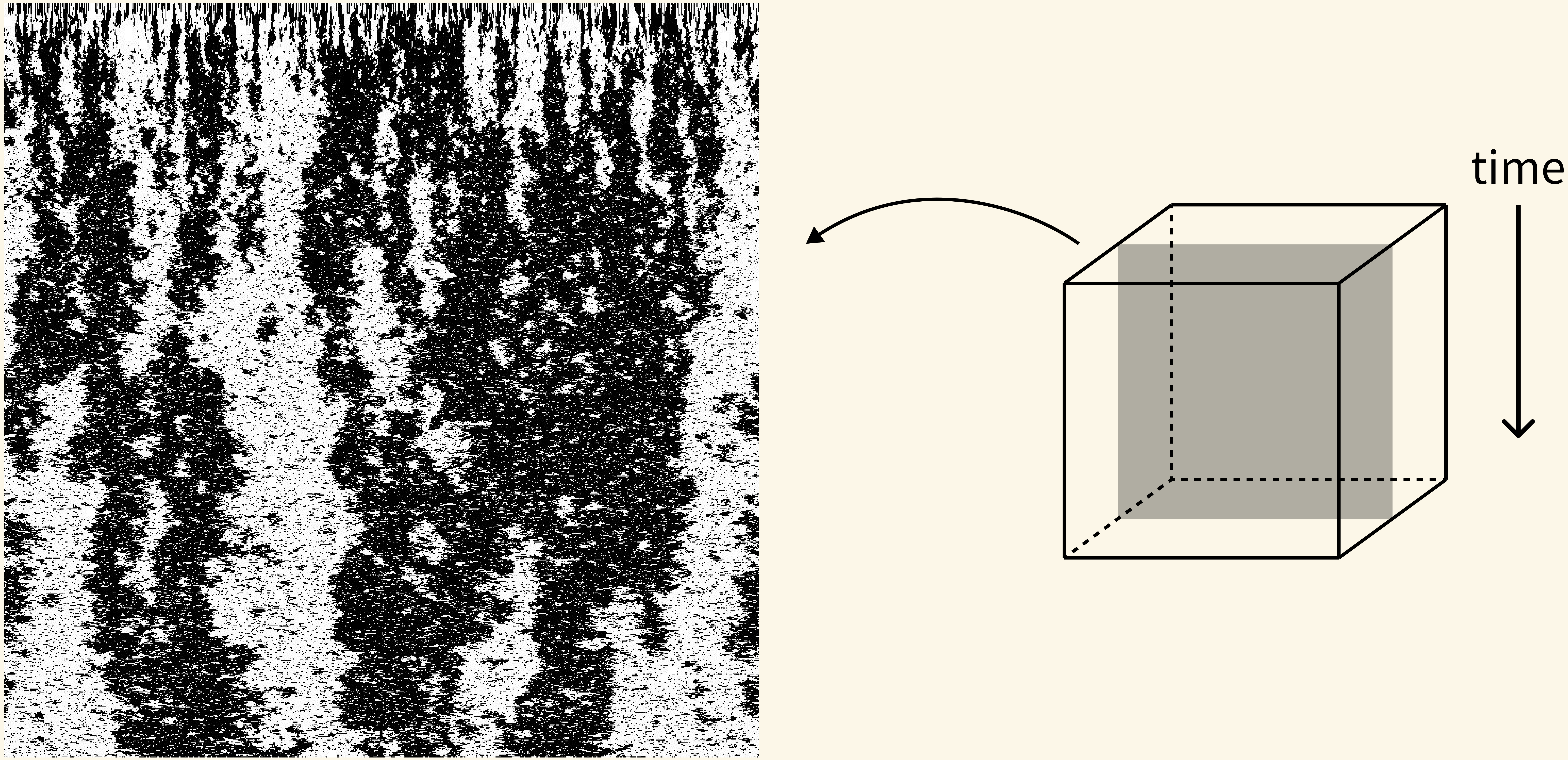
Temperature is fixed to the critical value



Construction of dynamics:

- Markov process $\dfrac{dp(\boldsymbol{\sigma})}{dt} = \sum_{\boldsymbol{\sigma}'} W_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} p(\boldsymbol{\sigma}')$
- Detailed balance $W_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} p_{\mathrm{eq}}(\boldsymbol{\sigma}') = W_{\boldsymbol{\sigma}'\boldsymbol{\sigma}} p_{\mathrm{eq}}(\boldsymbol{\sigma})$
- Locality (single spin flip)





Critical slowing down at critical points $\tau \sim L^{z}$ (τ : relaxation time, L : linear system size)

Numerical value of dynamical exponent: $z = 2.1667(5) \geq 2$ Nightingale, Blöte, PRB 62, 1089 (2000)

Prebious bound: $z \geq 7/4$ Halperin, PRB 8, 4437 (1973).
Lubetzky, Sly, Commun. Math. Phys. 313, 815 (2012).

Numerical values of z for several universality classes

Equilibrium states	dynamical exponent z
Classical Ising (2D)	$2.1667(5) \geq 2$
Classical Ising (3D)	$2.0245(15) \geq 2$
Classical Heisenberg (3D)	$2.033(5) \geq 2$
Three-state Potts (2D)	$2.193(5) \geq 2$
Four-state Potts (2D)	$2.296(5) \geq 2$

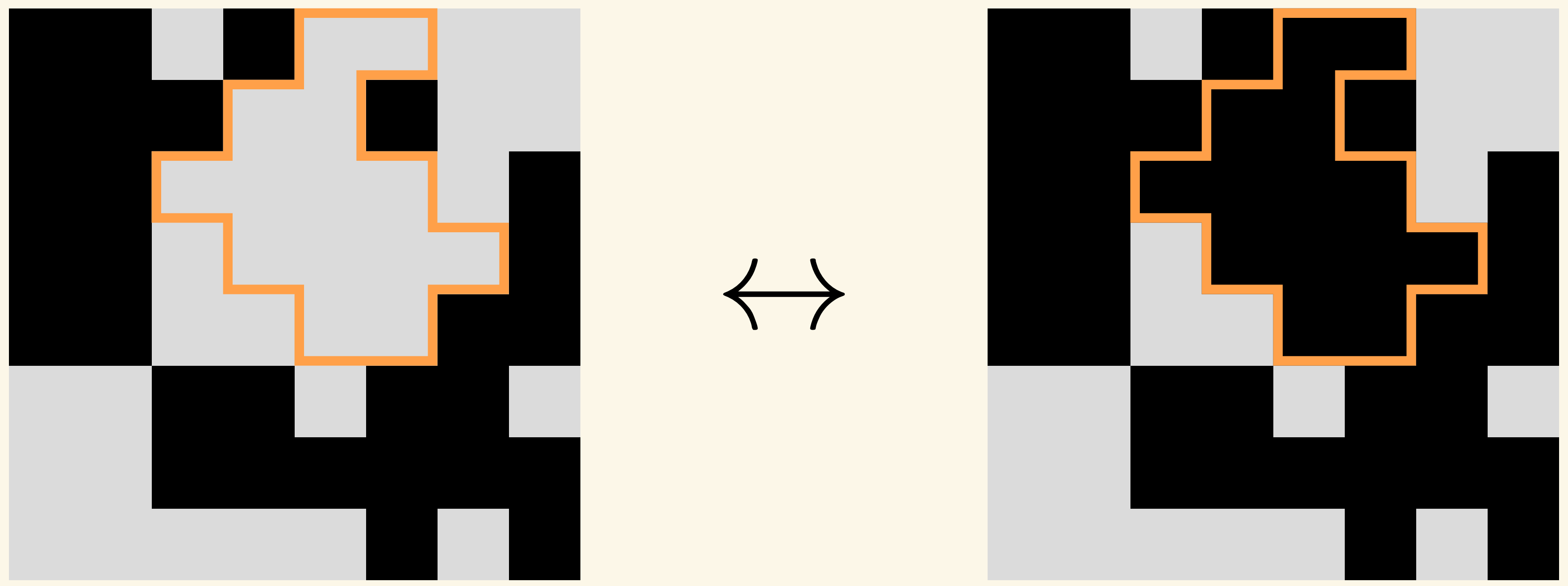
Implication of our result

No-go theorem for Markov chain Monte Carlo (MCMC) algorithms

Under the assumptions of detailed balance and local state updates, one can never achieve $z < 2$.

There has been a long-standing effort to develop faster MCMC algorithms, focusing on nonlocality or breaking detailed balance.

Swendsen, Wang, PRL 58, 86 (1987).
Wolff, PRL 62, 361 (1989).
Suwa, Todo, PRL 105, 120603 (2010).
Turitsyn et al., Physica D (Amsterdam) 240D, 410 (2011).



e.g. Wolff cluster algorithm $\rightarrow z \approx 0.3 < 2$

Our no-go theorem provides a rigorous foundation for these works!

Summary

- We first formulate unified conjectures on universal propeties of gapless FF systems.
- We provide several highly general proofs of our conjectures. (Still there is no complete proof)
- We found application of our results to classical stochastic systems.

Future course

- Refinement of proof
- Other universal properties of gapless FF systems
  Entanglement? Spatial conformal invariance? ← RM, arXiv:2511.16496 (2025).

Thank you for your attention!